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Image positions of a vertical rod in a liquid-filled cylindrical container

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Abstract

We describe a rather simple optical experiment, which many students can easily carry out, but the theoretical explanation of which requires far-from-simple mathematical analysis and application of numerical methods. A thin vertical rod, partially immersed in a liquid-filled transparent cylindrical container, is moved from the centre toward the wall and observed laterally. As it moves, the part seen through the air and the part seen through the water start to separate, so that at a certain distance from the centre the lower part of the rod becomes invisible. We show that this happens at the distance for which the refracted ray that reaches the observer's eye is along the tangent to the surface of a cylinder. We derive the expression for this distance as a function of the index of refraction and ratio d/r, where d is the distance of the observer from the surface and r is the radius of the container. The locus of image positions is determined by evaluating the intersections of pairs of close rays from the rod which reach the observer's eye after refraction at the surface of the container.

Keywords: refraction of light, index of refraction, negative index of refraction, visual phenomena due to the refraction of light, rays of light, measurement of the index of refraction

(Some figures may appear in colour only in the online journal)

1. Introduction

A variety of interesting optical phenomena are caused by the refraction and reflection of light at the interface of two media with different indexes of refraction [1–6]. One of the most popular demonstrations of refraction-related phenomena is to make visible a coin initially invisible in a bowl. The coin is invisible because the opaque wall of the bowl does not permit direct visual

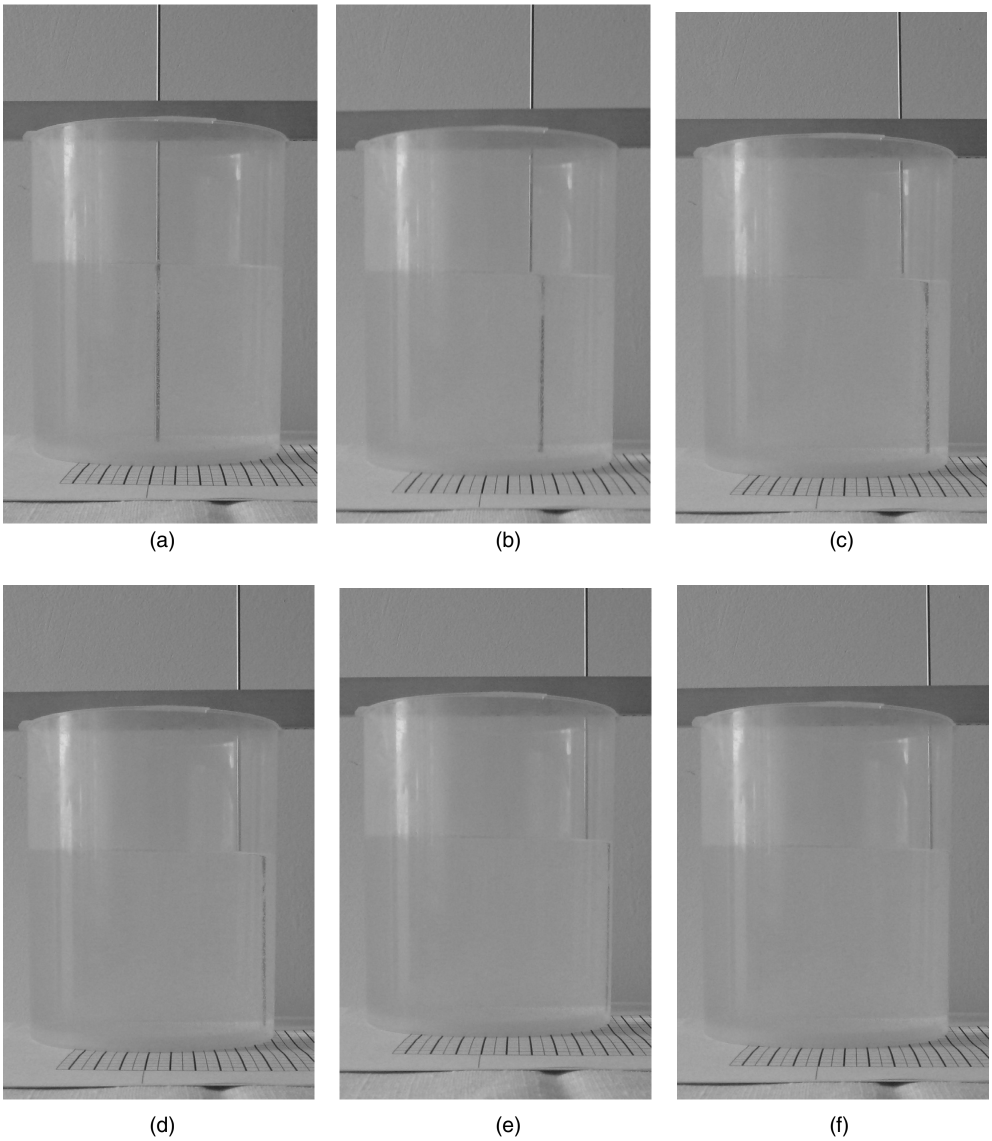

Figure 1. Photos of the experiment, with six distances of the rod from the centre: (a) $p = 0$; (b) $p = 20$ mm; (c) $p = 40$ mm; (d) $p = 45$ mm; (e) $p = 50$ mm; and (f) $p = 55$ mm. The radius of the container is $r = 6.35$ cm.

contact between the coin and the observer's eye. Pouring water gently into the bowl makes the invisible coin visible. This simple trick looks as 'magical' today as it did in ancient times. The first explanation of this phenomenon, based on the refraction of light, was elaborated by Ptolemy [1]. In this paper we consider an 'opposite' optical phenomenon in which something visible becomes invisible.

When a thin vertical rod which is partially immersed in water inside a transparent cylindrical container is placed at the centre of the cylinder (figure 1(a)), a lateral observer looking through the wall sees the rod as a complete object. The only difference between the upper part, seen through the air, and the lower one, seen through the water, is that the second is

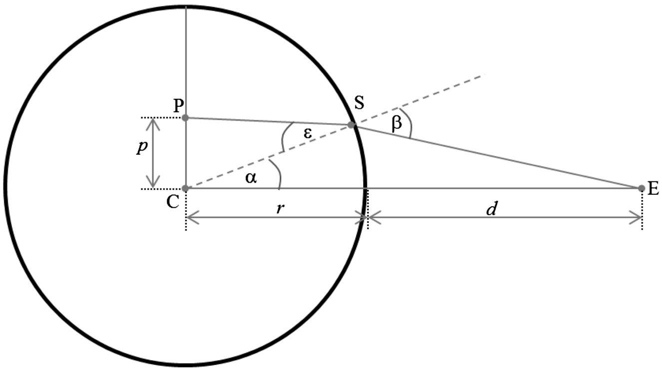

Figure 2. Horizontal cross section of the container filled with liquid. A ray from a rod (P) reaches the observer's eye E after refraction at S.

slightly wider. If the observer moves the rod away from the centre, the part seen through the air and the part seen through the water start to separate (figures 1(b)–(e)). The apparent position of the immersed part is further away from the centre than the position of the part in the air. Thus, the rod appears to be made of two separated parts. The distance between the lower and upper part of the rod increases as the rod is moved away from the centre. At a certain distance from the centre, the lower part of the rod becomes invisible or, in other words, it 'disappears' (figure 1(f)). Undoubtedly, many people may perceive the invisibility of the rod's immersed part as something amazing or even magical. Instead, physicists will try to find a conceptual model to explain the phenomenon qualitatively and then quantitatively.

Being acquainted with the laws of reflection and refraction of light, a physicist would conclude that the phenomenon is due to the refraction of light propagating from the rod to the observer's eye. As the rod moves to the right from the centre to the periphery of the container its 'depth decreases' because of the curvature of the liquid surface (figure 2). Due to this, the image moves slightly toward the observer. The image also moves further to the right from the rod because the point at the surface at which the light ray from the rod refracts depends on the rod's position. As the distance of the rod from the centre increases, its image eventually touches the wall (figure 1(e)). This happens when the refracted ray lies along a tangent to the surface (figure 3). With a further increase of the distance, the image of the lower part of the rod disappears (figure 1(f)). To determine the apparent positions of the lower parts of the rod quantitatively, a mathematical theory is needed.

Two refractions happen along the path of the light ray from the rod to the observer. The first happens at the liquid to glass interface and the second happens at the glass to air interface. But, if the wall of the container is thin, the refracted angle at the water–glass interface and the incident angle at the glass–air interface are the same. After applying the law of refraction twice, we see that the effect of the wall on light propagation may be neglected. Consequently, we will first analyse the case of a container with a thin wall and then in the appendix we consider the propagation of light taking into account the finite thickness of the wall.

In section 2, we derive the equation determining a point (S) at the surface of a cylinder in which a ray traveling from a rod (P) to the eye (E) is refracted. This equation is solved numerically for various values of the ratio p/r, where p denotes the distance of a rod form the centre and r is the radius of a container (section 3). A solution is presented graphically, together with graphs of other relevant quantities. Image positions determined using these solutions (section 4) are in agreement with the quantified visual observations. Approximate solutions for the positions of a rod near the centre of the container are given in section 5.

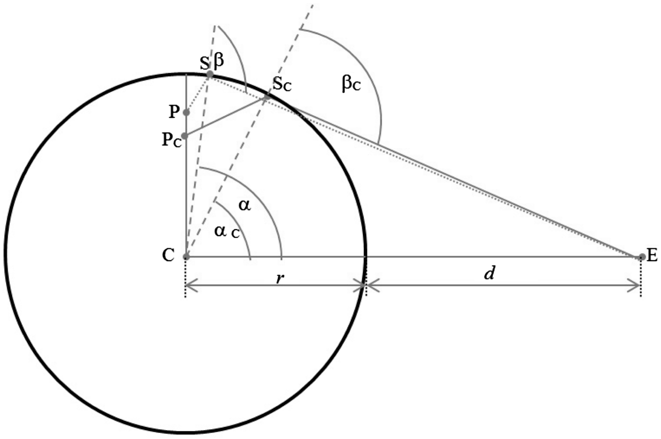

Figure 3. A scheme showing a particular property of point S_C, where the tangent at this point passes through the observer's eye. For positions P beyond P_C no ray from the rod could reach the observer's eye. The path PSE is associated with unphysical solutions of equation (11), such that $\alpha \geqslant \alpha_c$ and $\beta > \pi/2$. Formally, such solutions of equation (5) exist.

Approximate solutions are applicable to the measurement of the index of refraction of a liquid [5]. The effect of the wall on the propagation of light from the rod to the observer's eye is studied in the appendix. It is shown that, for a thickness of the glass of physical interest, one may replace the two refractions (liquid–glass and glass–air) by a single one between liquid and air.

2. An equation determining a point (S) on the surface of a cylinder in which a ray traveling from a rod (P) to the eye (E) is refracted

Let us consider a thin rod at a distance p from the centre C of a cylindrical container with radius r (figure 2), assuming that the thickness of the wall is negligible. An observer (E) at a distance d from the wall of the container observes the rod. According to Fermat's principle, or the principle of least time, the point S at the surface of the cylinder at which the ray is refracted will have a position such that the path PSE taken by a ray of light is the path where the travel time is stationary with respect to variations of that path [4]. This means that we should find the extreme of the function

$$T(\alpha) = \frac{1}{c}(n \cdot \overline{\mathrm{PS}} + \overline{\mathrm{SE}}) \tag{1}$$

where n is the index of refraction of the fluid in the container, $\alpha = \angle \mathrm{SCE}$, $\overline{\mathrm{PS}}$ and $\overline{\mathrm{SE}}$ are the lengths of the paths through liquid and air, respectively, which are given by :

$$\overline{\mathrm{PS}} = \sqrt{r^2 + p^2 - 2rp\sin\alpha}, \tag{2}$$

$$\overline{\mathrm{SE}} = \sqrt{(d+r)^2 + r^2 - 2r(r+d)\cos\alpha}. \tag{3}$$

By differentiating $T(\alpha)$ and equating it to zero

$$\frac{\mathrm{d}T(\alpha)}{\mathrm{d}\alpha} = \frac{1}{c}\left(n\frac{\mathrm{d}\overline{\mathrm{PS}}}{\mathrm{d}\alpha} + \frac{\mathrm{d}\overline{\mathrm{SE}}}{\mathrm{d}\alpha}\right) = 0 \tag{4}$$

one finds

$$-n\frac{p\cos\alpha}{\sqrt{r^2+p^2-2rp\sin\alpha}}+\frac{(r+d)\sin\alpha}{\sqrt{(d+r)^2+r^2-2r(r+d)\cos\alpha}}=0. \tag{5}$$

Another way to obtain the equation for angle α is to start from the law of refraction of light that connects the angle of incidence ε and the angle of refraction β:

$$n\sin\varepsilon=\sin\beta \quad \beta\leqslant 90^\circ. \tag{6}$$

We express $\sin\varepsilon$ and $\sin\beta$ in terms of p, r and d using the sinus theorem:

$$\frac{\overline{\mathrm{SE}}}{\sin\alpha}=\frac{r+d}{\sin\beta} \tag{7a}$$

$$\frac{p}{\sin\varepsilon}=\frac{\overline{\mathrm{PS}}}{\cos\alpha}. \tag{7b}$$

From (7*a*) and (7*b*) it follows

$$\sin\beta=(r+d)\frac{\sin\alpha}{\overline{\mathrm{SE}}} \tag{8a}$$

$$\sin\varepsilon=\frac{p}{\overline{\mathrm{PS}}}\cos\alpha. \tag{8b}$$

By substituting $\sin\beta$ and $\sin\varepsilon$ into (6), we get:

$$n\frac{p}{\overline{\mathrm{PS}}}\cos\alpha=(r+d)\frac{\sin\alpha}{\overline{\mathrm{SE}}}. \tag{9}$$

Using (2) and (3) we find:

$$n\frac{p}{\sqrt{r^2+p^2-2rp\sin\alpha}}\cos\alpha=(r+d)\frac{\sin\alpha}{\sqrt{(r+d)^2+r^2-2r(r+d)\cos\alpha}}. \tag{10}$$

We see that (10) is identical to (5). Therefore, Fermat's principle of stationary time and the refraction equation lead to the same equation for the angle α associated with a point S on the refracting surface.

For given values of r, p and d, one may determine from (5) the angle α and, consequently, the point S at the surface of a cylinder in which a ray of light traveling from the rod (P) to the eye (E) is refracted.

Of particular physical interest is the point S_C (see figure 3) for which $\beta\equiv\beta_c=\pi/2$. The corresponding distance, angle and position of the rod are denoted by p_c, α_c and P_C, respectively. By analysing the propagation of rays from the rod at positions such that $p>p_c$, one may conclude that no ray from these distances could reach the observer's eye. So, the lower part of the rod at positions beyond P_C is invisible.

As can be seen from figure 3, we have:

$$\cos\alpha_c=r/(r+d). \tag{11}$$

It follows from (6) that for the angle $\beta=\beta_c=90^\circ$ we have

$$\sin\varepsilon_c=1/n. \tag{12}$$

From the triangle CP_cS_c it follows that

$$\frac{p_c}{\sin\varepsilon_c}=\frac{r}{\cos(\alpha_c-\varepsilon_c)}. \tag{13}$$

From (12) and (13) it follows that

$$p_c=\frac{r}{n\cos(\alpha_c-\varepsilon_c)}. \tag{14}$$

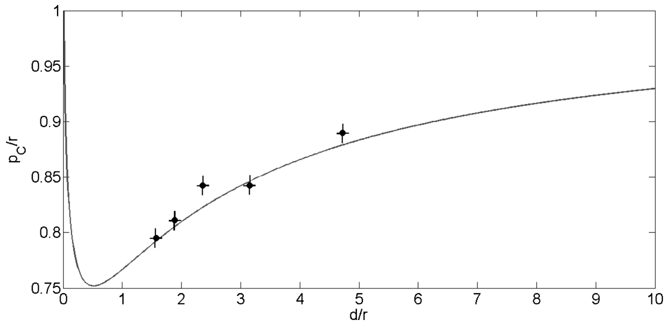

Figure 4. Relative critical distance p_c/r as a function of d/r for $n = 1.33$ (red curve) and measured values (black dots with error bars).

From the latter relation it follows that

$$p_c \geqslant \frac{r}{n}. \tag{15}$$

By substituting (11) into (5) one finds the explicit expression for relative distance p_c/r in terms of n and d/r:

$$\frac{p_c}{r} = \frac{1 + (d/r)}{\sqrt{n^2 - 1} + \sqrt{(d/r)(2 + (d/r))}}. \tag{16}$$

The dependence of relative critical distance on the ratio d/r for $n = 1.33$ is represented in figure 4. Several of the measured values of p_c/r, which were made by visually observing the disappearance of the lower part of the rod in the same container as at figure 1, are also given in figure 4. The agreement of theoretical and measured values is satisfactory. Experimental errors are due to uncertainties in measuring the distance of the viewing eye from the wall of the container ($\Delta d = 1$ cm) and uncertainties in reading the position of the rod ($\Delta p = 1$ mm).

Above, we have considered rays which leave the rod and reach the observer's eyes. In order to gain better insight into the phenomenon, it is useful to consider propagation of all of the other rays. In particular, it is interesting to investigate if there are rays that undergo total reflection. For this purpose, let us consider the incident angle ε of an arbitrary ray originated from the rod at the distance p from the centre C. From equations (2) and (8*b*) we find the values of the angle ε at various points on the surface of a liquid for the chosen position of the rod:

$$\varepsilon(\alpha) = \frac{p}{r} \frac{\cos\alpha}{\sqrt{1 + (p/r)^2 - 2(p/r)\sin\alpha}}. \tag{17}$$

Rays with incident angles $\varepsilon(\alpha) > 1/n$ will be totally reflected. The other rays, for which $\varepsilon(\alpha) < 1/n$, will be refracted. In this set of rays there will be a ray that will reach the observer's eye.

In figure 5, the plots of the difference $\varepsilon(\alpha) - \arcsin(1/n)$ for several values of the rod distance p are given.

By carefully analysing how plots change with p/r, we conclude that for $p < r/n$ all rays from the rod undergo refraction at the surface of a liquid and propagate to the air. Namely, the function $\varepsilon(\alpha) - \arcsin(1/n)$ is negative for all values of α. For $p \geqslant r/n$, the function $\varepsilon(\alpha) - \arcsin(1/n)$ is positive in a certain interval of α and is negative for all other values of α. This means that there are two sets of rays. In the first set are the rays which undergo refraction and in the second are the rays which undergo total reflection.

The role of the distance $p = r/n$ is clearly understood by looking to figure 6. Since $\angle P_1A_1C < \angle EDC$, $\angle P_1B_1C < \angle EDC$ and $\angle P_1C_1C < \angle EDC$, all rays emerging from P_1 are

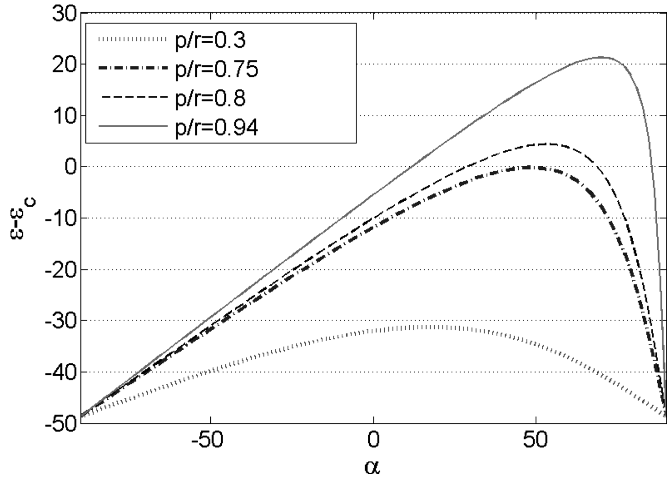

Figure 5. Graphs of the function $\varepsilon - \varepsilon_c = \varepsilon(\alpha) - \arcsin(1/n)$ for four values of the relative rod distance p/r.

refracted and propagate into the air. The incident angle of ray P_2B_2 is chosen to be equal to the critical angle. It is seen that $\angle P_2A_2C < \angle P_2B_2C < \angle P_2C_2C$ so the ray P_2C_2 is totally reflected while the ray P_2A_2 propagates into the air.

3. Solution of the equation for the refraction point at the surface

In this section we shall determine the solutions of equation (5) numerically by finding the zeros of the function

$$F(X) = -n\frac{p}{r}X\sqrt{\left(1+\frac{d}{r}\right)^2 + 1 - 2\left(1+\frac{d}{r}\right)X} + \left(1+\frac{d}{r}\right)\sqrt{1-X^2}\sqrt{1+\left(\frac{p}{r}\right)^2 - 2\frac{p}{r}\sqrt{1-X^2}} \tag{18}$$

where $X = \cos\alpha$. Before presenting the solutions, let us analyse some of the properties of these solutions.

Equation (5) always has solutions in the interval $X \in [0, 1]$ since $F(0) > 0$ and $F(1) < 0$. But, as follows from the analysis in the previous section, only the solutions such that $\alpha \leqslant \alpha_c$ are of physical interest. Since the cosine function is decreasing on the interval $(0, \pi/2)$, the solutions for X which are of physical interest are in the range

$$1 \geqslant X \geqslant r/(r+d) \equiv X_c = \cos\alpha_c. \tag{19}$$

The corresponding range of β is $0 \leqslant \beta \leqslant 90^\circ$.

The solutions such that $\alpha > \alpha_C$, i.e. $0 \leqslant X \leqslant r/(r+d)$, and $\beta > \pi/2$ are not of physical interest because part of the straight line SE would lie inside the container (figure 3), implying that after a point S the ray would partly travel through the liquid. This means that the corresponding propagation of light would not be consistent with the initial physical model.

After determining the values of $X = \cos\alpha$ and α for given values of n, p/r and d/r, one determines angles β and ε using the relations (2), (3) and (8), which lead to:

$$\varepsilon = \arcsin\left(p\frac{\cos\alpha}{\sqrt{r^2+p^2-2rp\sin\alpha}}\right) \quad \text{for} \quad \alpha < \alpha_c \tag{20}$$

$$\beta = \arcsin(n\sin\varepsilon) \quad \text{for} \quad \alpha < \alpha_c. \tag{21}$$

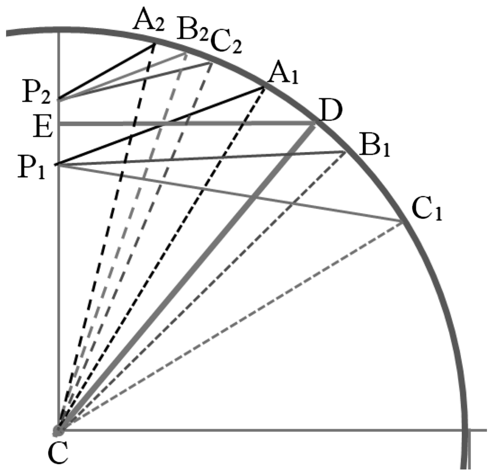

Figure 6. Geometric analysis of incident angles to the outer surface of the container for two characteristic positions of the rod: P_1 ($\overline{CP_1} < \overline{CE} = r/n$) and P_2 ($\overline{CP_2} > \overline{CE} = r/n$). Note that $\angle EDC = \arcsin(1/n)$ is equal to the critical incidence angle for the total internal reflection ε_C.

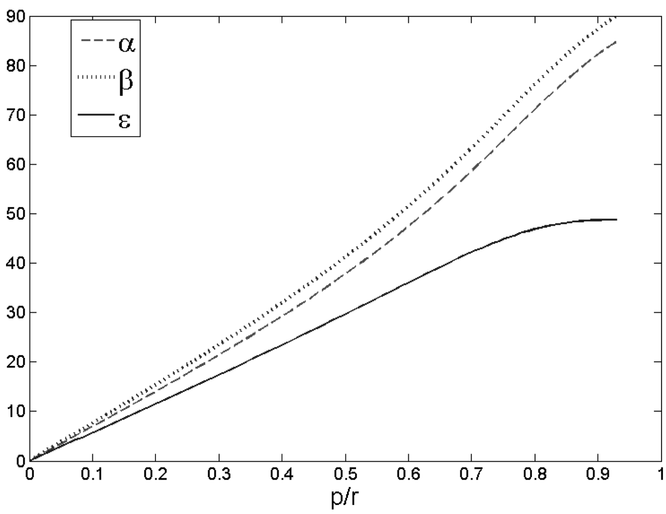

Figure 7. Graphs of solutions of angles α, β and ε as functions of the relative rod distance p/r for $p/r < p_c/r = 0.93$ and $d = 10\ r$.

The dependence of angles α, β and ε on p/r obtained numerically for $n = 1.33$ and $d/r = 10$ are shown in figure 7.

4. The determination of the position of the image

To determine the image of a rod it is necessary to consider two close rays travelling from a rod to the eye along close paths PSE and PS′E′. The position of S is determined by solving equation (5), while S′ is close to S, which is chosen so that the vertical separation of two rays at the place of the observer is much smaller than the diameter of the pupil of the eye. The

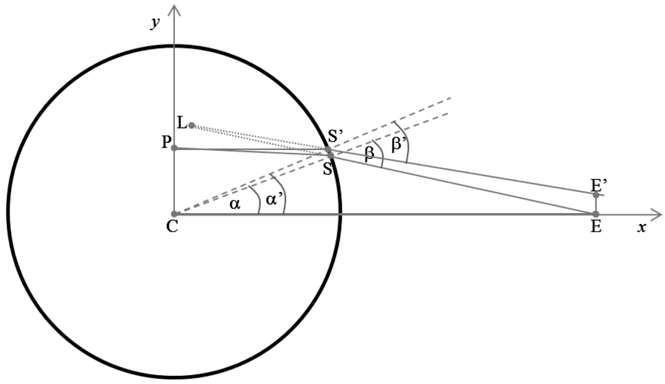

Figure 8. Scheme of the intersection of two close rays.

image is formed at the intersection of the prolongation of parts ES and E′S′ of these close paths, at point L, as represented at figure 8.

The equations of the straight lines passing through points S and E, and S′ and E′, in the reference frame shown in figure 8, are:

$$y = r\sin\alpha - \tan(\beta - \alpha)(x - r\cos\alpha), \tag{22}$$

$$y = r\sin\alpha' - \tan(\beta' - \alpha')(x - r\cos\alpha'). \tag{23}$$

The coordinates of the point of intersection of these lines, i.e. the coordinates of the image, are:

$$x_L = \frac{r\sin\alpha - r\sin\alpha' + \tan(\beta - \alpha)r\cos\alpha - \tan(\beta' - \alpha')r\cos\alpha'}{\tan(\beta - \alpha) - \tan(\beta' - \alpha')}, \tag{24}$$

$$y_L = r\sin\alpha - \tan(\beta - \alpha)(x_L - r\cos\alpha). \tag{25}$$

Using the procedure above described and the solutions of (5) given above, the positions of the images of a rod for seven positions of a rod are represented in figure 9 and in table 1.

5. Useful approximations near the centre of the container

From the solutions for angles α, β and ε presented in figure 7, and the coordinates of the images presented at figure 9, we see that near the centre of the container the images lie approximately along the line that connects the centre and the position of a rod.

As can be seen from figure 10, the following approximation near the centre ((p/r) <0.3) is valid:

$$x_L \cong 0, \quad y_L \cong np. \tag{26}$$

Applying this approximation, one may determine the index of refraction of a liquid as proposed by Gluck [5]. For this purpose, Gluck considers a vertical thick rod, with a diameter d_r, in a cylindrical container filled with liquid. The relations that we derived above for an infinitesimally thin rod are applied to the edges of a thick rod.

One moves a rod from the centre, toward the right, until the right edge of a rod above the liquid coincides with the left edge of its image, as shown in figure 11. Let us denote this characteristic distance of the left edge of the rod from the centre by p_k. By applying

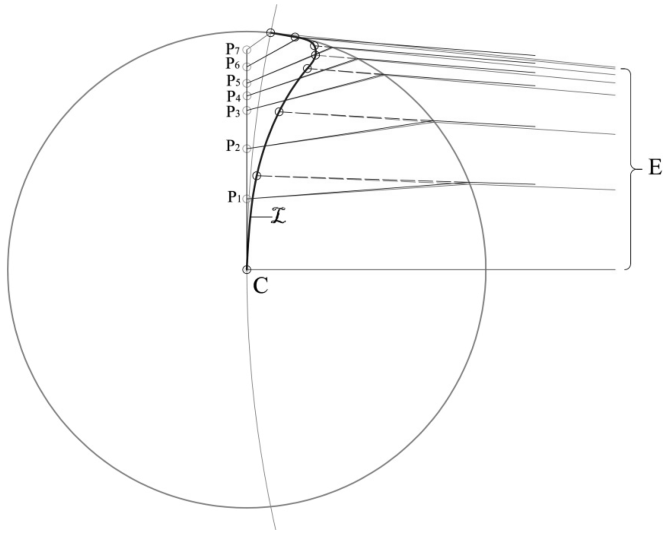

Figure 9. Positions of a rod and its image for seven distances of the rod from the centre. The observer's distance is $d = 10r$, $r = 60$ mm and $n = 1.33$.

Table 1. Coordinates of the rod images for seven positions of the rod.

P_i	p_i (mm)	x_L (mm)	y_L (mm)
P_1	17.85	2.49	23.66
P_2	30.48	8.10	39.77
P_3	40.11	15.19	50.67
P_4	43.77	17.25	54.03
P_5	46.94	16.96	56.40
P_6	51.09	12.12	58.60
P_7	55.40	6.00	59.69

approximation (26) and taking into account the definition of the distance p_k, we may write

$$y_k = p_k + d_r = np_k. \tag{27}$$

From the latter relation it follows that,

$$n = \frac{p_k + d_r}{p_k}. \tag{28}$$

Therefore, by measuring the distance p_k defined above, one determines from the relation (28) the index of refraction of a liquid. This is a simple method proposed by Gluck without justification of approximate relations (26). In this work, we show that these approximate relations are valid near the centre ((p/r) <0.3) of a container. Since detachment of the image from the rod happens at the distance $p_k = d_r/(n-1)$, it follows from this consideration that a measurement should be performed with a rod with the diameter $d_r < 0.3 \cdot (n-1) \cdot r$.

The measurement error of the index of refraction with this method is:

$$\Delta n = \frac{\Delta d_r}{p_k} + \frac{d_r}{p_k^2}\Delta p_k. \tag{29}$$

For water, Gluck [5] obtained for the index of refraction of water the value $n = 1.35$. Taking $\Delta d_r = 0.1$ mm and $\Delta p_k = 2$ mm the relative error of the result is 3%.

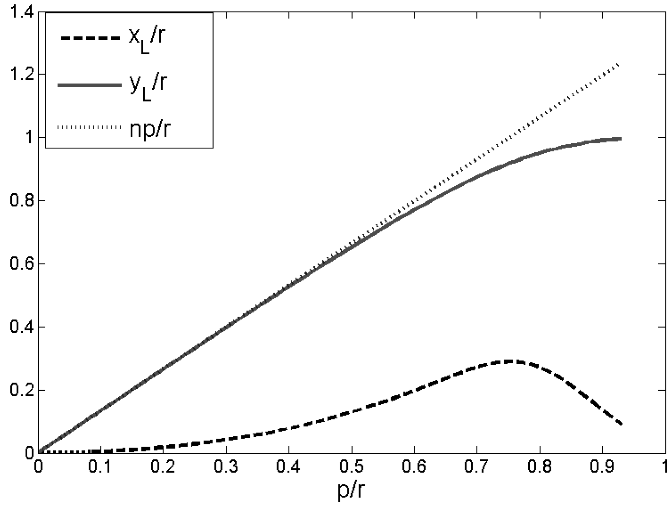

Figure 10. Graphs of relative coordinates x_L/r and y_L/r of the image, as functions of the relative rod distance, for $p/r < p_c/r = 0.93$ and $d = 10\ r$. The blue dotted line represents function np/r.

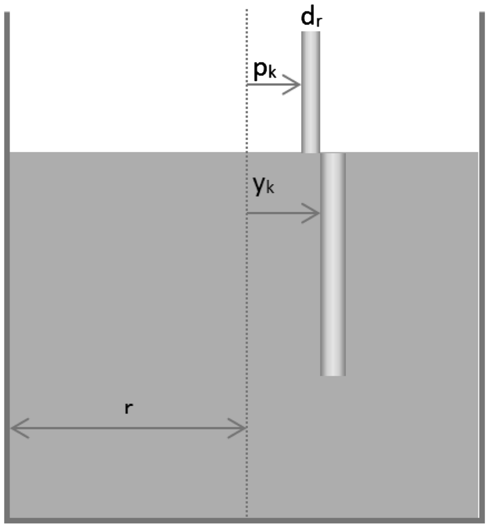

Figure 11. Sketch of the experiment with a thick rod near the centre of a container, at the position where the right edge of the rod is at the same distance from the centre as the left edge of its image.

6. Conclusions

Despite the simplicity and availability of the experiment with a moving vertical rod in a cylindrical container filled with transparent liquid, its theoretical explanation and the determination of the image positions are not at all simple. From the physical side, one uses the law of refraction of light. The physics of vision requires us to consider two neighbouring rays from the object (rod) and to determine the intersection of prolongation of the refracted parts of these two rays. The geometry of the problem requires us to use and combine several mathematical theorems, leading to the equation for the position of the point at the surface of the cylinder at which rays arriving at the observer's eye are refracted (equation (5)). For the general position of the rod, this equation is not solvable analytically and it has to be solved using numerical methods. Nevertheless, the critical distance of the rod after which the lower part of the rod cannot be seen, equation (16), is determined analytically from equation (5). The influence of the wall is also studied by considering the propagation of rays, taking into account the refraction at the liquid–wall and wall–air interfaces. It is shown that the loci of image positions, evaluated by taking and by not taking into account the presence of the wall, almost coincide.

Approximate relations, valid for positions of the rod near the centre of the container, may be used to determine the index of refraction of a liquid.

Thus, the beauty and usefulness of the experiment described here for educational purposes are its simplicity and availability, as well as the wealth of knowledge and skills to be demonstrated in its conceptual and quantitative explanation.

Acknowledgments

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Appendix. The influence of the wall glass on the image positions

Due to the presence of the glass wall, there are two refractions along the path of light propagation, which are at the liquid to glass and at the glass to air interfaces (figure A1). We shall denote by S_1 and S the points of refraction at the internal and external glass surfaces, by r_1 and r_2 the internal and external radii of the container, and by n_1 and n_2 the index of refraction of the liquid and glass, respectively. The meaning of the other symbols in figure A1 are self-explanatory.

In this case, it is more complicated to determine the positions of the rod images for various positions of the rod. Fortunately, there are two methods, a direct method and an inverse method, the inverse method being simpler than the direct method.

In the direct method one generalizes the method described in section 2. Namely, instead of looking for extremes of the function (1), one looks for the extreme of the function:

$$T_1(\alpha) = \frac{1}{c}(n_1 \cdot \overline{\mathrm{PS}_1} + n_2 \cdot \overline{\mathrm{SS}_1} + \overline{\mathrm{SE}}) \tag{A.1}$$

where

$$\overline{\mathrm{PS}_1} = \sqrt{r_1^2 + p^2 - 2r_1 p \sin\alpha_1} \tag{A.2a}$$

$$\overline{\mathrm{SE}} = \sqrt{(d + r_2)^2 + r_2^2 - 2r_2(r_2 + d)\cos\alpha} \tag{A.2b}$$

$$\overline{\mathrm{SS}_1} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\alpha_1 - \alpha)} \tag{A.2c}$$

and the angle α_1 depends on the angle α. This dependence is found by applying the law of refraction at two interfaces:

$$n_1 \sin\varepsilon = n_2 \sin\delta \tag{A.3a}$$

$$n_2\ \sin\delta_1 = \sin\beta \quad \beta \leqslant 90^\circ \tag{A.3b}$$

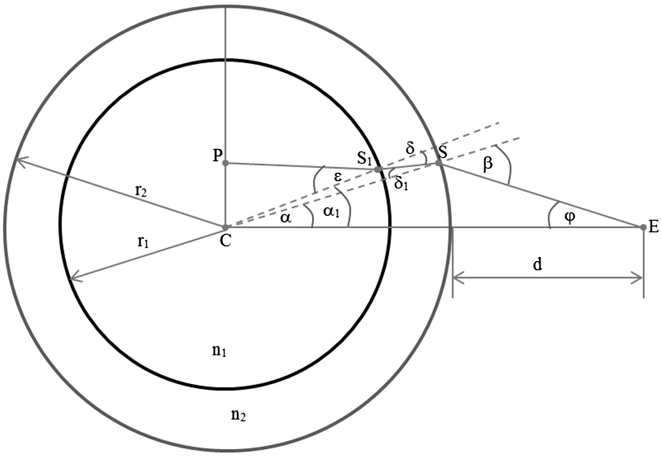

Figure A1. A ray from a rod (P) reaches the observer's eye E after two refractions, at the liquid–glass interface (point S_1) and at the glass–air interface (point S).

and the geometrical relations between angles and radii r_1 and r_2 at figure A1:

$$\alpha = \beta - \varphi \tag{A.4a}$$

$$\alpha_1 = \alpha + \delta - \delta_1 \tag{A.4b}$$

$$\frac{r_1}{\sin\delta_1} = \frac{r_2}{\sin\delta}. \tag{A.4c}$$

In the inverse method, one uses the relations (A.3) and (A.4) to find the position of the rod for the given positions of the observer's eye and point S at the external surface of the wall. So, in this case, the independent variable is the angle φ. The task is to determine the distance p of the rod from the centre of the container. Using relations (A.3) and (A.4), one at first expresses all angles in terms of the variable angle φ and parameters r_1, r_2, n_1, n_2:

$$\beta(\varphi) = \arcsin\left(\frac{r_2 + d}{r_2} \cdot \sin\varphi\right) \tag{A.5a}$$

$$\alpha(\varphi) = \beta(\varphi) - \varphi \tag{A.5b}$$

$$\delta_1(\varphi) = \arcsin\left(\frac{\sin\beta(\varphi)}{n_2}\right) \tag{A.5c}$$

$$\delta(\varphi) = \arcsin\left(\frac{r_2}{r_1} \cdot \sin\delta_1(\varphi)\right) \tag{A.5d}$$

$$\alpha_1(\varphi) = \alpha(\varphi) + \delta(\varphi) - \delta_1(\varphi) \tag{A.5e}$$

$$\varepsilon(\varphi) = \arcsin\left(\frac{n_2}{n_1} \cdot \sin\delta(\varphi)\right). \tag{A.5f}$$

Now, from the triangle $\overline{\mathrm{CS_1P}}$, one finds:

$$p(\varphi) = r_1 \cdot \sin\alpha_1(\varphi) + r_1 \cdot \cos\alpha_1(\varphi) \cdot \tan[\varepsilon(\varphi) - \alpha_1(\varphi)]. \tag{A.6}$$

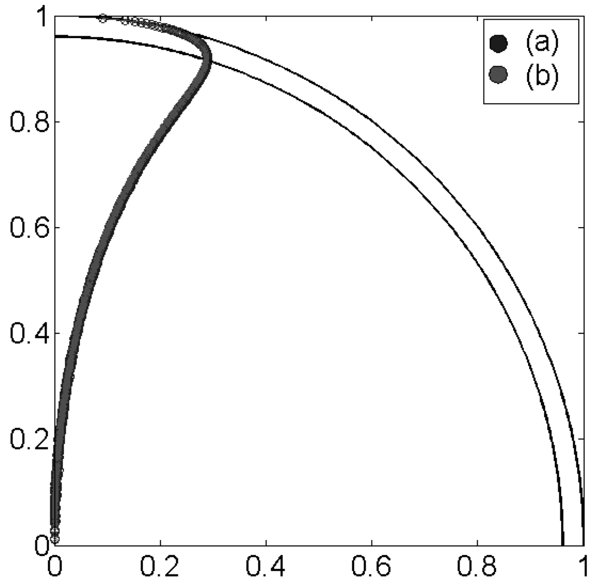

Figure A2. Locus of images of a rod in two cases: (a) the wall of the container is infinitesimally thin—only refraction at the liquid–air interface is considered; and, (b) the wall of the container is thick—refraction at both the liquid–glass and glass–air interfaces is considered. $n_1 = 1.33$, $n_2 = 1.55$, $d = 10\ r_2$, $r_1 = 0.96\ r_2$.

Now, using the obtained dependence $p(\varphi)$ for one chosen ray (angle φ), one determines the intersection of this ray and another ray that is very close to it. By repeating this procedure for angles $\varphi \in [0, a\tan(r_2/(d + r_2))]$, one obtains the red curve in figure A2.

From figure A2, one clearly sees that the difference between image positions evaluated by taking and not taking into account the presence of the wall is very small. In our evaluation, the ratio of the wall thickness and radius of the container is $(r_2 - r_1)/r_2 = 0.04$. In our experiment this ratio is even smaller, namely 0.016.

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